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Magnetic brane solutions in AdS

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ABSTRACT: We construct asymptotically AdS₅ solutions of Einstein-Maxwell theory dual to $\mathcal{N} = 4$ SYM theory on $\mathbb{R}^{3,1}$ in the presence of a background magnetic field. The solutions interpolate between AdS₅ and a near horizon AdS₃ × T². The central charge of the near horizon region, and hence low temperature entropy of the solution, is found to be $\sqrt{\frac{4}{3}}$ times that of free $\mathcal{N} = 4$ SYM theory. The entropy vanishes at zero temperature. We also present the generalization of these solutions to arbitrary spacetime dimensionality.

KEYWORDS: Black Holes in String Theory, AdS-CFT Correspondence

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1 Introduction

In the AdS/CFT correspondence boundary gauge theories in the presence of background electromagnetic fields can be studied by imposing suitable boundary conditions on gauge fields in the bulk of AdS. By turning on such fields one can compute the electrical conductivity of the gauge theory, study its response to magnetic fields, and so forth.

The solution for a black brane in AdS₄ with a magnetic field is easily found and has many AdS/CFT applications (e.g [1]–[11]) to the study of 2 + 1 gauge theories in magnetic fields. It is clearly of interest to have such solutions in the AdS₅ case as well. For instance, as recently emphasized in [12], strongly coupled gauge theories in magnetic fields arise at RHIC, and one would like to be able to use holography to study the effect of the magnetic field. To the extent that 3 + 1 dimensional condensed matter systems can be modeled by AdS/CFT, the ability to turn on magnetic fields provides a valuable probe of the system with a clear physical meaning.

Given this motivation, in this paper we find magnetic brane solutions to five dimensional Einstein-Maxwell theory with a negative cosmological constant. The most important properties of these solutions can be determined analytically, although some numerical work is needed to capture all the details.

The solutions that we find are related to, but distinct from, previously studied “AdS₅ black string” solutions (and their higher dimensional cousins) [13]–[18], as well as the solutions of Maldacena and Nunez [19]. In these papers, two of the spatial direction are

taken to be compact (usually S^2 or H^2) and the magnetic field strength is taken to be proportional to the curvature two-form, as is in fact required by supersymmetry. On the other hand, we will be looking for solutions with a nonzero field strength on a flat boundary metric. Our solutions are thus intrinsically non-supersymmetric.

Taking the spatial boundary directions to be a compact torus, our solutions interpolate between $\text{AdS}_3 \times T^2$ at small r and AdS_5 at large r . At finite temperature the AdS_3 factor is replaced by a BTZ black hole [20]. The black brane entropy density correspondingly interpolates between a linear T dependence at low temperature and a T^3 dependence at high temperature.

We then compare our results to the thermodynamics of free $\mathcal{N} = 4$ SYM theory in an external $U(1)_R$ magnetic field. At high temperature we recover the standard result $S_{\text{grav}} = \frac{3}{4} S_{\mathcal{N}=4}$. At low temperature the $\mathcal{N} = 4$ theory reduces to a 1 + 1 dimensional conformal field theory described by the fermion zero modes, with a central charge equal to N^2 times the number of units of quantized magnetic flux. The corresponding strong coupling result can be computed in gravity from the Brown-Henneaux formula, and we find an increase by a factor of $\sqrt{\frac{4}{3}}$, i.e. $c_{\text{grav}} = \sqrt{\frac{4}{3}} c_{\mathcal{N}=4}$. The low temperature entropy computed from gravity therefore is also enhanced by this factor compared to the free $\mathcal{N} = 4$ result. It is amusing that turning on a magnetic field actually improves the numerical agreement between gravity and free $\mathcal{N} = 4$ SYM, giving a relative $\sqrt{\frac{4}{3}}$ versus $\frac{3}{4}$.

The property that the entropy of the AdS_5 magnetic brane solution vanishes at zero temperature distinguishes it from the AdS_4 case, where there is a finite entropy at extremality. In fact, this property also matches what one would expect from a free fermion description. In the AdS_4 case all spatial directions of the boundary are threaded by magnetic flux; there is thus a finite density of fermion zero modes per unit area, and hence a finite entropy density.

It is straightforward to extend our considerations to arbitrary spacetime dimension. For odd dimensional AdS_{d+1} spacetimes and maximal rank magnetic field, we find solutions interpolating between $\text{AdS}_3 \times T^{d-2}$ and AdS_{d+1} , with properties very similar to the AdS_5 case. Similarly, for even dimensional AdS_{d+1} spacetimes the situation parallels the AdS_4 case; explicit solutions are easily found, and the solutions have a finite entropy extremal limit.

This paper is organized as follows. In section 2 we construct magnetic brane solutions in AdS_5 . In section 3 we compute the thermodynamics of free $\mathcal{N} = 4$ SYM theory in a background magnetic field. In section 4 we generalize to arbitrary spacetime dimension. Some conclusions are given in section 5. The appendix details our progress in looking for analytic solutions of the AdS_5 equations.

2 Magnetic brane in AdS_5

The action of five-dimensional Einstein-Maxwell theory with a negative cosmological constant is¹

$$S = -\frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left(R + F^{MN} F_{MN} - \frac{12}{L^2} \right) + S_{\text{bndy}} . \quad (2.1)$$

¹Conventions: $R^\lambda_{\mu\nu\kappa} = \partial_\kappa \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\kappa} + \Gamma^\eta_{\mu\nu} \Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa} \Gamma^\lambda_{\nu\eta}$ and $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$.

The boundary terms include the Gibbons-Hawking term as well as other contributions necessary for a well posed variational principle [21, 22]; their explicit forms will not be needed here. Along with the Bianchi identity, the field equations are

$$R_{MN} = \frac{4}{L^2}g_{MN} + \frac{1}{3}F^{PQ}F_{PQ}g_{MN} - 2F_{MP}F_N^P \quad (2.2)$$

$$\nabla^M F_{MN} = 0 .$$

We henceforth set the AdS radius to unity: $L = 1$.

Were we to add to the action the Chern-Simons term

$$S_{CS} = \frac{k}{16\pi G_5} \int A \wedge F \wedge F, \quad k = \frac{8}{3\sqrt{3}} \quad (2.3)$$

then our action would correspond to the bosonic part of $D = 5$ minimal gauged supergravity (e.g. [23]). The Chern-Simons term makes no contribution to the solutions considered in this paper, but will be useful for fixing the normalization of the gauge field.

We are interested in solutions asymptotic to AdS_5 with a magnetic field tangent to the boundary directions. The field strength and metric can be taken to be invariant under spacetime translations, rotations in the $x^{1,2}$ plane, and time reversal. A general ansatz consistent with the symmetries is

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + e^{2V(r)}\left((dx^1)^2 + (dx^2)^2\right) + e^{2W(r)}dy^2 \quad (2.4)$$

$$F = Bdx^1 \wedge dx^2$$

The Maxwell equation is automatically satisfied, and the Einstein equations reduce to

$$U(V'' - W'') + \left(U' + U(2V' + W')\right)(V' - W') = -2B^2e^{-4V} \quad (2.5)$$

$$2V'' + W'' + 2(V')^2 + (W')^2 = 0$$

$$\frac{1}{2}U'' + \frac{1}{2}U'(2V' + W') = 4 + \frac{2}{3}B^2e^{-4V}$$

$$2U'V' + U'W' + 2U(V')^2 + 4UV'W' = 12 - 2e^{-4V}B^2$$

With r as the evolution parameter, the final equation represents a constraint on initial data. Once imposed on the initial data, it is automatically satisfied on the full solution by virtue of the three dynamical equations. Alternatively, one can omit one of the dynamical equations, since it will be implied by the remaining dynamical equations together with the derivative of the constraint equation.

With $B = 0$, a solution to these equations is AdS_5 , represented by $U = e^{2V} = e^{2W} = r^2$. For nonzero B an exact solution is given by $U = 3(r^2 - r_+^2)$, $e^{2V} = B/\sqrt{3}$, $e^{2W} = 3r^2$. This solution represents the product of a BTZ black hole and T^2 (we are taking the $x^{1,2}$ directions to be compact),

$$ds^2 = -3(r^2 - r_+^2)dt^2 + \frac{dr^2}{3(r^2 - r_+^2)} + \frac{B}{\sqrt{3}}\left((dx^1)^2 + (dx^2)^2\right) + 3r^2dy^2 . \quad (2.6)$$

Our goal is to find solutions that interpolate between (2.6) at small r and AdS₅ at large r . From the boundary field theory point of view this represents an RG flow between a $D = 3 + 1$ CFT at short distance and a $D = 1 + 1$ CFT at long distance. As we discuss in the next section, this is the expected behavior of $\mathcal{N} = 4$ SYM in the presence of uniform external magnetic flux.

The central charge of the near horizon AdS₃ region can be computed from the Brown-Henneaux formula [24], $c = 3l/(2G_3)$. The AdS₃ radius is $l = 1/\sqrt{3}$, Taking $x^{1,2}$ to be compact with coordinate volume V_2 , the $D = 3$ Newton constant is $G_3 = \sqrt{3}G_5/(BV_2)$. Using the AdS₅/CFT₄ relation $G_5 = \pi/(2N^2)$, we find

$$c = \frac{N^2 BV_2}{\pi} = \sqrt{\frac{4}{3}} \left(\frac{BV_2}{2\pi} \right) N^2 . \tag{2.7}$$

In the last step we have written the result in a form convenient for comparison with $\mathcal{N} = 4$ SYM theory, where the rescaled magnetic field is $\mathcal{B} = \sqrt{3}B$. The combination $BV_2/(2\pi)$ will be identified with the number of quantized units of magnetic flux. Knowledge of the central charge (2.7) will be sufficient information to deduce the low temperature behavior of the black hole entropy.

We have not succeeded in solving (2.5) analytically to find the interpolating solutions, but numerical integration is straightforward. The procedure is a bit different depending on whether the temperature is zero or nonzero.

2.1 Zero temperature solutions

At zero temperature we can look for solutions preserving Lorentz invariance in the (t, y) directions, which corresponds to setting $U = e^{2W}$. The system of equations (2.5) reduces to

$$\begin{aligned} 2V'' + W'' + 2(V')^2 + (W')^2 &= 0 \\ V'^2 + W'^2 + 4V'W' &= 6e^{-2W} - e^{-4V-2W} B^2 . \end{aligned} \tag{2.8}$$

Note that B can be absorbed by a shift of V . Starting from the small r behavior $e^{2V} = B/\sqrt{3}$ and $e^{2W} = 3r^2$, we numerically integrate out and find a solution with large r behavior $e^{2V} = vr^2$ and $e^{2W} = r^2$. Numerically, we find $v \approx 1.87B$. The result is a smooth zero temperature solution interpolating between a near horizon AdS₃ \times T^2 and an asymptotic AdS₅. There is a unique such solution, inasmuch as the value of B can be absorbed by a rescaling of $x^{1,2}$.

2.2 Finite temperature solutions

Now we turn to the finite temperature solutions. The solutions carry a nonzero temperature and magnetic field; however, using the freedom to rescale coordinates there is really only a one parameter family of solutions, which we can think of as being parameterized by the dimensionless combination $\frac{T}{\sqrt{B}}$. The numerical analysis proceeds as follows. We rescale r such that the horizon is at $r = 1$; i.e. $U(1) = 0$. By rescaling t we can take $U'(1) = 1$, which sets the temperature to a fixed value, leaving B as the free parameter. Further, by rescaling $x^{1,2}$ and y we can take $V(1) = W(1) = 0$. With these conditions, the first and

third equations of (2.5) give us the initial data $V'(1) = 4 - \frac{4}{3}b^2$ and $W'(1) = 4 + \frac{2}{3}b^2$, where we are writing b for the value of the magnetic field in these coordinates. We then integrate out to find solutions with asymptotic behavior $U = r^2$, $e^{2V} = vr^2$, and $e^{2W} = wr^2$, where v and w are functions of the free parameter b , to be computed numerically. We find smooth solutions for all values of b , thus exhibiting the existence of solutions interpolating between a near horizon BTZ $\times T^2$ and an asymptotic AdS₅.

The solution has a conformal boundary metric $ds^2 = -dt^2 + v((dx^1)^2 + (dx^2)^2) + wdy^2$. To put this in standard form we can introduce the coordinates $\tilde{x}^{1,2} = \sqrt{v}x^{1,2}$ and $\tilde{y} = \sqrt{w}y$. The Hawking temperature, determined from the imaginary time periodicity, is $T = 1/(4\pi)$. Since the field strength takes the form

$$F = bdx^1 \wedge dx^2 = \frac{b}{v}d\tilde{x}^1 \wedge d\tilde{x}^2, \tag{2.9}$$

it is $B = b/v$ that represents the physical magnetic field. Similarly, the physical entropy density is

$$\frac{S}{V} = \frac{1}{4G_5} \frac{1}{v\sqrt{w}}. \tag{2.10}$$

It is most illuminating to display the numerical results as a plot of the entropy density versus temperature. Since it is only dimensionless quantities that are meaningful, we divide each by the appropriate power of the magnetic field B . Or rather since it is $\mathcal{B} = \sqrt{3}B$ that will appear naturally on the field theory side, we divide by powers of \mathcal{B} . For the temperature, we thus compute

$$\frac{T}{\sqrt{\mathcal{B}}} = \frac{3^{-1/4}}{4\pi} \sqrt{\frac{v}{b}}. \tag{2.11}$$

For the entropy density, using $G_5 = \frac{\pi}{2N^2}$ and dividing through by N^2 we compute

$$\frac{S}{VN^2\mathcal{B}^{3/2}} = \frac{3^{-3/4}}{2\pi} \sqrt{\frac{v}{b^3w}}. \tag{2.12}$$

The numerical results, together with those of the field theory computation discussed in the next section, are shown in figure 1.

The behavior of the black brane entropy at high and low temperatures can be determined analytically. At high temperatures the magnetic field becomes a subleading effect, and we recover the standard result for finite temperature D3-branes, namely a T^3 dependence with the entropy being 3/4 of that obtained from the free field limit of $\mathcal{N} = 4$ SYM theory with gauge group $U(N)$.

At low temperature we have a BTZ black hole. The entropy of a BTZ black hole (or more generally any system with the symmetries of a $D = 1 + 1$ CFT) is given by $S = \frac{\pi}{3}cTL_y$, where we are taking the y coordinate to be compact, and where in general c is the average of the left and right moving central charges. One might be worried that since g_{yy} is a nontrivial function of r , the relevant size of the y circle differs depending on whether we measure it in the BTZ region or at infinity in the AdS₅ region. The same

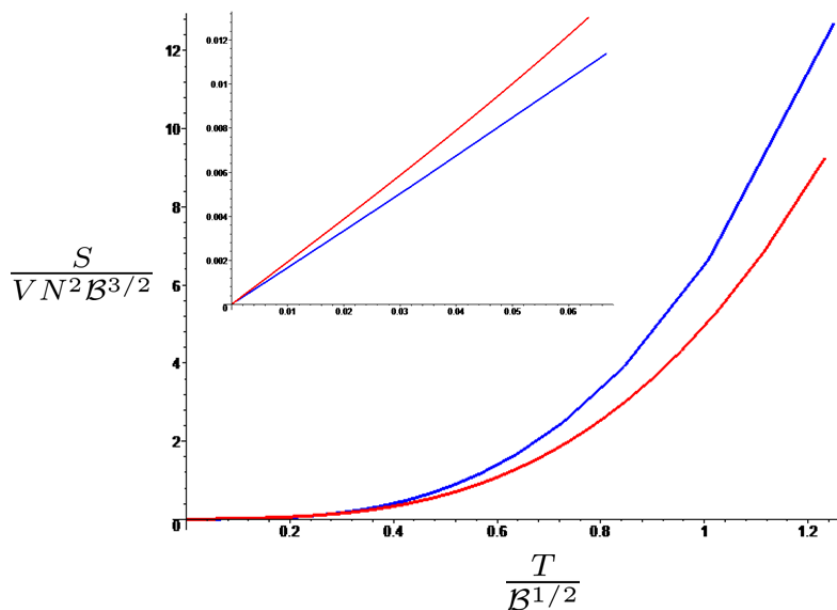


Figure 1. Plot of entropy versus temperature for gravity (red) and free $\mathcal{N} = 4$ SYM theory (blue). The inset shows the low temperature behavior. At low temperature the entropies are linear in T , with $S_{\text{grav}} = \sqrt{\frac{4}{3}}S_{\mathcal{N}=4}$. At high temperature the entropies are cubic in T , with $S_{\text{grav}} = \frac{3}{4}S_{\mathcal{N}=4}$. Gravity gives the larger entropy at low temperature by a factor of $\sqrt{\frac{4}{3}}$; as the temperature is raised the curves cross, and then asymptotically the gravitational entropy is lower by a factor of $\frac{3}{4}$.

is true of the time coordinate, which determines the temperature. Fortunately, due to the zero temperature condition $U = e^{2W}$, the two possible rescalings cancel, so that the product TL_y is the same in the BTZ region as at infinity. Using the result (2.7) the low temperature entropy becomes

$$\frac{S}{V} = \frac{N^2}{3\sqrt{3}}BT . \tag{2.13}$$

We have verified that (2.13) indeed matches the low temperature numerics, as can be seen from the inset in figure 1.

The numerics show that the entropy smoothly interpolates between the linear and cubic in T dependence as the solution interpolates between a BTZ black hole and a five dimensional black brane.

3 Comparison with $\mathcal{N} = 4$ Super Yang-Mills

We now work out the entropy of the free field limit of $\mathcal{N} = 4$ SYM theory in an external magnetic field in order to compare with the black brane entropy. Our first task is to fix the normalization of the field theory magnetic field relative to that used on the gravity side.

We should first state more specifically which magnetic field is under consideration. On the gravity side, we stated that if we include the term (2.3) then the action corresponds to minimal gauged supergravity. This implies that the bulk gauge field appears on the boundary as an external field coupled to the R-symmetry current of the field theory. Here we are thinking of the $\mathcal{N} = 4$ theory in $\mathcal{N} = 1$ terms, with a U(1) R-symmetry. The natural normalization in the field theory is to assign the gaugino R-charge 1. In this normalization, the field content of the $\mathcal{N} = 4$ theory is as follows: we have N^2 Weyl spinors of charge 1 (the gauginos); $3N^2$ Weyl spinors of charge $-\frac{1}{3}$ (the fermions in the chiral multiplets); $3N^2$ complex scalar fields of charge $\frac{2}{3}$ (the scalars in the chiral multiplets); and N^2 vector fields of charge 0 (the gauge fields).

To fix the relative normalizations we compare the anomalous variations under gauge transformations of the R-symmetry gauge fields, $\delta\mathcal{A} = d\tilde{\Lambda}$. On the field theory side we use the standard result from the triangle anomaly,

$$\delta S_{\text{eff}} = \frac{1}{24\pi^2} \text{tr} Q^3 \int \tilde{\Lambda} \mathcal{F} \wedge \mathcal{F}, \tag{3.1}$$

where the trace is over the spectrum of Weyl fermions. In our case, $\text{tr} Q^3 = \frac{8N^2}{9}$, and so

$$\delta S_{\text{eff}} = \frac{N^2}{27\pi^2} \int \tilde{\Lambda} \mathcal{F} \wedge \mathcal{F}. \tag{3.2}$$

On the gravity side the anomalous variation comes from the Chern-Simons term (2.3), whose coefficient was fixed by supersymmetry. This gives

$$\delta S = \frac{k}{16\pi G_5} \int \Lambda F \wedge F = \frac{N^2}{3^{3/2}\pi^2} \int \Lambda F \wedge F. \tag{3.3}$$

Comparing, we find the relation $\mathcal{F} = \sqrt{3}F$, which is the result that we used in the previous section.

We now compute the partition function, $Z = \text{Tr} e^{-\beta H}$, of $\mathcal{N} = 4$ SYM theory at finite temperature and magnetic field, at zero coupling. From the partition function we extract the entropy using the standard formula $S = (1 - \beta \frac{\partial}{\partial \beta}) \ln Z$, and then compare with the black brane entropy. For $\mathcal{B} = 0$ it is well known that the two entropies only differ by a factor of $\frac{3}{4}$, even though the field theory and gravity computations are valid in the non-overlapping regimes of small and large 't Hooft coupling. It is interesting to extend this comparison to nonzero magnetic field; as we'll see, this actually improves the agreement between the entropies.

In the free field limit, to compute Z we only need to know the spectrum of single particle excitations in the presence of a magnetic field pointing along the y direction, which are given by relativistic Landau levels. First consider a charge q_ϕ scalar field. Solving $D^\mu D_\mu \phi = 0$, we find the energies

$$E = \pm \sqrt{p_y^2 + (2n + 1)|q_\phi \mathcal{B}|}, \quad n = 0, 1, 2, \dots \tag{3.4}$$

As usual, the branch with $E < 0$ corresponds to charge $-q_\phi$ antiparticles with positive energy. Each mode has a degeneracy corresponding to the number of units of magnetic

flux, $|q_\phi \mathcal{B}|V_2/(2\pi)$, where V_2 denotes the area in the $x^{1,2}$ plane transverse to $\vec{\mathcal{B}}$. Summing over both the particles and anti-particles, the scalar partition function is

$$\ln Z_\phi(q_\phi) = -2 \frac{|q_\phi \mathcal{B}|V_2}{2\pi} \sum_{n=0}^{\infty} \frac{L_y}{2\pi} \int_{-\infty}^{\infty} dp_y \ln \left(1 - e^{-\beta \sqrt{p_y^2 + |q_\phi \mathcal{B}|(2n+1)}} \right). \quad (3.5)$$

Next, consider a charge q_ψ Weyl spinor. We solve $\gamma^\mu D_\mu \psi = 0$ subject to $\gamma^5 \psi = \psi$. This yields the following spectrum of energies. First, there are solutions obeying $E^2 > p_y^2$ with

$$E = \pm \sqrt{p_y^2 + 2|q_\psi \mathcal{B}|n}, \quad n = 1, 2, \dots \quad (3.6)$$

Second, there are solutions with $E^2 = p_y^2$. For $q_\psi \mathcal{B} > 0$ these obey $E = p_y$; for $q_\psi \mathcal{B} < 0$ they obey $E = -p_y$. These are zero modes of the two-dimensional Dirac operator, whose existence is mandated by the index theorem. The sign of the momentum of the physical $n = 0$ excitations is correlated with the sign of q_ψ . All of the solutions have degeneracy $|q_\psi \mathcal{B}|V_2/(2\pi)$. The partition function of a charge q_ψ Weyl spinor is thus²

$$\ln Z_\psi(q_\psi) = \frac{|q_\psi \mathcal{B}|V_2}{2\pi} \sum_{n=0}^{\infty} \sum_{\alpha=\pm 1} \frac{L_y}{2\pi} \int_{-\infty}^{\infty} dp_y \ln \left(1 + e^{-\beta \sqrt{p_y^2 + |q_\psi \mathcal{B}|(2n+1-\alpha)}} \right). \quad (3.7)$$

The $\alpha = 1$ part includes the zero mode contribution.

The gauge fields are neutral, and have partition function

$$\ln Z_V = -2 \frac{V_2 L_y}{(2\pi)^3} \int d^3 p \ln \left(1 - e^{-\beta |p|} \right). \quad (3.8)$$

The total partition function corresponding to the field content of $\mathcal{N} = 4$ SYM is

$$\ln Z = N^2 \left(3 \ln Z_\phi(2/3) + \ln Z_\psi(1) + 3 \ln Z_\psi(-1/3) + \ln Z_V \right). \quad (3.9)$$

In the high temperature limit the sums over n can be replaced by integrals, and we recover the standard result for the entropy,

$$\frac{S}{V} = \frac{2\pi^2}{45} \left(g_b + \frac{7}{8} g_f \right) T^3 \quad (3.10)$$

where $g_{b,f}$ denote the number of bosonic/fermionic helicity states. In the present context, $g_b = g_f = 8N^2$.

In the low temperature limit the partition function is dominated by the fermionic zero mode contribution. From (3.7) we see that each $D = 3 + 1$ fermion contributes the same as $|q_\psi \mathcal{B}|V_2/(2\pi)$ fermions in $D = 1 + 1$, with corresponding central charge $c = |q_\psi \mathcal{B}|V_2/(4\pi)$. The low temperature entropy is thus

$$S \approx \frac{\pi}{3} c L_y T, \quad c = \sum_{\psi} \frac{1}{2} \frac{|q_\psi \mathcal{B}|V_2}{2\pi} = \frac{|\mathcal{B}|V_2}{2\pi} N^2. \quad (3.11)$$

²For a given sign of q_ψ the zero mode fermions really have a definite sign of p_y , but to simplify (3.7) we replace the integration over anti-particles of $p_y > 0$ (say) by the equivalent integration over $p_y < 0$.

Comparing with (2.7) we see that the central charges, and hence the low temperature entropies, differ by a factor of $\sqrt{\frac{4}{3}}$. Somewhat surprisingly, the strong coupling result coming from gravity gives the larger central charge, in contrast to the result that at high temperature the gravitational entropy is less by a factor of 3/4 compared to free $\mathcal{N} = 4$ SYM theory.

For intermediate values of the temperature we evaluate the sums and integrals numerically, and obtain the result displayed in figure 1. Both the gravity and field theory entropies smoothly interpolate between a linear and cubic temperature dependence, corresponding to the fact that both are interpolating between $D = 1 + 1$ and $D = 3 + 1$ CFTs.

4 Generalization to AdS_{d+1}

In this section we discuss black brane solutions with magnetic fields in arbitrary dimensions. There are two basic cases, depending on whether the spacetime dimensionality is odd or even. The odd dimensional case, with maximal rank B-field, is analogous to the solutions constructed in section 2. The solutions interpolate between a near horizon AdS₃ and an asymptotic AdS_{d+1}. The solutions in the even dimensional case are analogous to the magnetic black brane in AdS₄ (see [25] for related solutions). These black branes have an extremal limit with nonzero entropy density, and a corresponding near horizon AdS₂ factor. For both odd and even dimensions, the low temperature behavior of the entropy matches that of massless free fermions in magnetic fields.

More generally, by considering B-fields of less than maximal rank, one can have solutions interpolating between a near horizon AdS_{d+1-2r} and an asymptotic AdS_{d+1}, where r is a positive integer.

In $d + 1$ dimensions the Einstein-Maxwell equations are

$$R_{MN} = \frac{d}{L^2}g_{MN} + \frac{1}{d-1}F^{PQ}F_{PQ}g_{MN} - 2F_{MP}F_N^P \quad (4.1)$$

$$\nabla^M F_{MN} = 0$$

There is also the Bianchi identity. For clarity, we restore the L -dependence in this section.

4.1 d odd

We choose a field strength of maximal rank. By rotating and scaling coordinates we can always take $F_{12} = F_{34} = \dots = B$. The metric ansatz is

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + e^{2V(r)}\left((dx^1)^2 + \dots + (dx^{d-1})^2\right). \quad (4.2)$$

The Einstein equations become

$$U'' + (d-1)U'V' = \frac{2d}{L^2} + 2B^2e^{-4V} \quad (4.3)$$

$$U'' + 2(d-1)\left(UV'' + UV'^2 + \frac{U'V'}{2}\right) = \frac{2d}{L^2} + 2B^2e^{-4V}$$

$$UV'' + U'V' + (d-1)UV'^2 = \frac{d}{L^2} - B^2e^{-4V}$$

A solution is given by

$$U = \frac{r^2}{L^2} + \left(\frac{L^4}{4-d} \right) \frac{B^2}{r^2} - \frac{M}{r^{d-2}}, \quad e^{2V} = \frac{r^2}{L^2}. \quad (4.4)$$

This is a magnetic black brane, with a horizon at $U(r_+) = 0$.

The Hawking temperature is

$$T = \frac{1}{4\pi} U'(r_+) = \frac{1}{4\pi} \left(\frac{dr_+}{L^2} - \frac{L^4 B^2}{r_+^3} \right). \quad (4.5)$$

The extremal limit is thus $r_+^2 = \frac{L^3 B}{\sqrt{d}}$.

The entropy density is

$$\frac{S}{V} = \frac{1}{4G_{d+1}} \left(\frac{r_+}{L} \right)^{d-1} \quad (4.6)$$

which at extremality becomes

$$\frac{S}{V} = \frac{1}{4G_{d+1}} \left(\frac{LB}{\sqrt{d}} \right)^{\frac{d-1}{2}}. \quad (4.7)$$

This entropy is proportional to the number of zero modes of massless fermions on T^{d-1} in the presence of magnetic flux.

The energy density can be worked out by integrating $E = \int TdS$ at fixed B , and gives

$$\frac{E}{V} = \frac{(d-1)}{16\pi G_{d+1}} \frac{M}{L^{d-1}}. \quad (4.8)$$

At extremality this becomes

$$\frac{E}{V} = \frac{1}{4\pi G_{d+1}} \left(\frac{d-1}{4-d} \right) \left(\frac{LB}{\sqrt{d}} \right)^{\frac{d}{2}} \frac{1}{L}. \quad (4.9)$$

Surprisingly, this is negative for $d > 4$. This does not imply an instability in the theory, since it is not meaningful to compare this energy against that for $B = 0$, as the geometries have different asymptotics.

4.2 d even

We let the field strength fill the directions x^1, x^2, \dots, x^{d-2} and denote the “left over” spatial direction by y . By rotating and scaling coordinates we can always take $F_{12} = F_{34} = \dots = B$ and $F_{iy} = 0$. The metric ansatz is

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + e^{2V(r)} \left((dx^1)^2 + \dots + (dx^{d-2})^2 \right) + e^{2W(r)} dy^2. \quad (4.10)$$

The Einstein equations can be reduced to

$$\begin{aligned} U(V'' - W'') + \left(U' + U \left((d-2)V' + W' \right) \right) (V' - W') &= -2B^2 e^{-4V} \\ (d-2)(V'' + V'^2) + W'' + W'^2 &= 0 \\ (d-2) \left(U'V' + 2UV'W' + (d-3)UV'^2 \right) + U'W' &= \frac{d(d-1)}{L^2} - (d-2)B^2 e^{-4V} \end{aligned} \quad (4.11)$$

These equations admit a BTZ $\times T^{d-2}$ solution

$$ds^2 = -(d-1) \left(\frac{r^2 - r_+^2}{L^2} \right) dt^2 + \frac{1}{(d-1)} \left(\frac{L^2}{r^2 - r_+^2} \right) dr^2 + (d-1) \frac{r^2}{L^2} dy^2 \quad (4.12)$$

$$+ \frac{LB}{\sqrt{d-1}} \left((dx^1)^2 + \dots + (dx^{d-2})^2 \right).$$

The AdS₃ radius is $l = \frac{L}{\sqrt{d-1}}$. The $D = 3$ Newton constant is

$$\frac{1}{G_3} = \left(\frac{LB}{\sqrt{d-1}} \right)^{\frac{d-2}{2}} V_{d-2} \frac{1}{G_{d+1}} \quad (4.13)$$

where V_{d-2} denotes the coordinate volume. The Brown-Henneaux central charge is

$$c = \frac{3l}{2G_3} = \frac{3}{2} \frac{1}{\sqrt{d-1}} \left(\frac{B}{\sqrt{d-1}L} \right)^{\frac{d-2}{2}} V_{d-2} \frac{L^{d-1}}{G_{d+1}} \quad (4.14)$$

This central charge is proportional to that which would arise from massless fermions on T^{d-2} in the presence of magnetic flux. The low temperature entropy is

$$S \approx \frac{\pi}{3} c T L_y = \frac{\pi}{2} \left(\frac{B}{\sqrt{d-1}L} \right)^{\frac{d-2}{2}} \frac{L^{d-1}}{G_{d+1}} \frac{T}{\sqrt{d-1}} V_{d-1}. \quad (4.15)$$

The interpolating solutions can be found by numerical integration of (4.11).

5 Discussion

In this work we have constructed magnetic brane solutions in AdS, using a combination of analytical and numerical methods. We mainly focussed on the AdS₅ case, and noted that the black brane thermodynamics agrees surprisingly well with that of free $\mathcal{N} = 4$ SYM theory. Although we framed our discussion in terms of $\mathcal{N} = 4$ SYM theory, it is worth noting that our solutions apply equally well to the much larger class of $\mathcal{N} = 1$ superconformal field theories that have a gravity dual described by Einstein-Maxwell theory with a negative cosmological constant. As shown in [26–28] this class includes all such theories with a dual IIB or M-theory description, as all these theories admit a consistent truncation to $D = 5$ minimal gauged supergravity.

A natural extension of this work is to add nonzero charge and momentum density to these solutions [29]. This leads to AdS₃ being replaced by a charged, rotating, BTZ black hole. The addition of a charge density leads to a nonzero current flow; this can be seen in $\mathcal{N} = 4$ SYM theory from the fact that there is a left-right asymmetry in the effective $D = 1 + 1$ CFT. This is in turn related to the triangle anomaly, and presumably gives a microscopic explanation for some of the effects noted in [12].

It would be interesting to study the transport properties of these solutions. For that purpose, it clearly would be desirable to have an analytic solution available, and in the appendix we report on our efforts in that direction.

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A Towards exact solutions

It would clearly be valuable to obtain exact analytic solutions to the reduced equations (2.5) for the general magnetic brane, and for their boost-invariant zero temperature limit in (2.8). The knowledge of an analytic solution greatly facilitates the study, for example, of the spectrum and dynamics of small fluctuations. Thus far, we have not succeeded in completely solving either one of these equations in all generality.

In this appendix, we shall present two partial analytic solutions of (2.5) and (2.8). The first consists of the regime of large magnetic field, specifically when the B^2e^{-4V} terms dominate the constant terms on the rhs of (2.5); this problem will be solved completely analytically. The second consists of the zero temperature regime, where boost invariance in the x^3 direction can be assumed; this problem will be reduced, by quadratures, to the solutions of a single first order ODE. So far, we have not succeeded in solving this remaining ODE. In this appendix, we shall provide the derivations of these analytic results.

A.1 The large B case

When B^2e^{-4V} is large, we ignore the constant terms on the rhs of (2.5). The fourth equation is the constraint, which is r -independent in view of the first three equations, and may be enforced as an initial datum. Using the constraint equation, we may eliminate the B^2e^{-4V} term from the first three equations. The resulting equations involve only the functions u, v, w , defined by (the sign and numerical factors are for later convenience),

$$u \equiv -U'/(6U) \qquad v \equiv -V'/3 \qquad w \equiv -W'/3 \qquad (\text{A.1})$$

and not the actual functions U, V, W , and become,

$$\begin{aligned} u' &= 6u^2 + 10uv + 5uw + 2v^2 + 4vw \\ v' &= -2uv - 4uw + 2v^2 - 5vw \\ w' &= 4uv + 8uw + 2v^2 + 3w^2 + 10vw \end{aligned} \qquad (\text{A.2})$$

In the two independent ratios u'/v' and w'/v' , the derivative with respect to r is effectively traded in for a derivative with respect to v . The right hand sides are homogeneous in u, v, w of degree 0, and may be expressed solely in terms of the ratios,

$$\alpha \equiv u/v \qquad \beta \equiv w/v \qquad (\text{A.3})$$

and we obtain,

$$\begin{aligned} v \frac{d\alpha}{dv} &= \frac{4\alpha^2\beta + 8\alpha^2 + 10\alpha\beta + 8\alpha + 4\beta + 2}{-4\alpha\beta - 2\alpha - 5\beta + 2} \\ v \frac{d\beta}{dv} &= \frac{4\alpha\beta^2 + 8\beta^2 + 10\alpha\beta + 4\alpha + 8\beta + 2}{-4\alpha\beta - 2\alpha - 5\beta + 2} \end{aligned} \qquad (\text{A.4})$$

Taking the ratio of these equations in turn gives an ordinary first order differential equation,

$$\frac{d\beta}{d\alpha} = \frac{4\alpha\beta^2 + 8\beta^2 + 10\alpha\beta + 4\alpha + 8\beta + 2}{4\alpha^2\beta + 8\alpha^2 + 10\alpha\beta + 8\alpha + 4\beta + 2} \quad (\text{A.5})$$

Equation (A.5) may be solved by considering the ratio $d(\alpha - \beta)/d(\alpha + \beta) = (\alpha - \beta)/\lambda$ in terms of the independent variable $\lambda \equiv \alpha + \beta + 1$. The general solution is given by,

$$\alpha - \beta = c\lambda \quad (\text{A.6})$$

where c is an arbitrary real integration constant. Using this solution back into (A.4), we derive v by quadrature, and by using (A.3), we find u and w as well,

$$\begin{aligned} u(\lambda) &= \frac{1}{2}v_0 \left\{ (1+c)\lambda - 1 \right\} \lambda^{-3/2} (\lambda - \lambda_+)^{\gamma_+} (\lambda - \lambda_-)^{\gamma_-} \\ v(\lambda) &= v_0 \lambda^{-3/2} (\lambda - \lambda_+)^{\gamma_+} (\lambda - \lambda_-)^{\gamma_-} \\ w(\lambda) &= \frac{1}{2}v_0 \left\{ (1-c)\lambda - 1 \right\} \lambda^{-3/2} (\lambda - \lambda_+)^{\gamma_+} (\lambda - \lambda_-)^{\gamma_-} \end{aligned} \quad (\text{A.7})$$

where v_0 is a real integration constant, and the constants λ_{\pm} and γ_{\pm} are given by,

$$\begin{aligned} \lambda_{\pm} &\equiv \frac{-3 \pm \sqrt{12 - 3c^2}}{1 - c^2} \\ \gamma_{\pm} &= \pm \frac{3(c+3) + 3\lambda_{\pm}^{-1}}{4\sqrt{12 - 3c^2}} \end{aligned} \quad (\text{A.8})$$

Finally, we use (A.2) to obtain also r in terms of λ , and we find,

$$v_0(1 - c^2)r(\lambda) = \int_{\lambda_0}^{\lambda} d\lambda' (\lambda')^{\frac{1}{2}} (\lambda' - \lambda_+)^{-1-\gamma_+} (\lambda' - \lambda_-)^{-1-\gamma_-} \quad (\text{A.9})$$

where λ_0 is a real integration constant. For the special case of the boost invariant solution, for which $c = 0$, the data work out as follows, $\lambda_{\pm} = -3 \pm 2\sqrt{3}$, and $4\gamma_{\pm} = \pm 2\sqrt{3} + 1$.

A.2 The boost-invariant case

Boost invariance in the x^3 -direction requires $U = 3e^{2W}$, and reduces the equations (2.5) to,

$$\begin{aligned} 2V'' + 2(V')^2 + W'' + (W')^2 &= 0 \\ V'' - W'' + 2(V')^2 - 3(W')^2 + V'W' &= -\frac{2}{3}B^2e^{-4V-2W} \\ (V')^2 + (W')^2 + 4V'W' &= \frac{2}{L^2}e^{-2W} - \frac{1}{3}B^2e^{-4V-2W} \end{aligned} \quad (\text{A.10})$$

It is readily checked that the r -derivative of the constraint vanishes in view of the first two equations of (A.10). Thus, the constraint may again be imposed as initial conditions. To solve the system, we concentrate of the first two equations of (A.10).

Taking the derivative of the second equation, and eliminating the B^2e^{-4V-2W} between the original equation and its derivative gives an equation that only involves V' and W' and its derivatives, but not the original fields V, W . We introduce again the notations of (A.1)

for v and w , eliminate w'' using the derivative of the first equation in (A.10), so that the final two independent equations become,

$$\begin{aligned} 2v' + w' &= 6v^2 + 3w^2 \\ -v'' + 8vv' - 4ww' + v'w + vw' &= 3(4v + 2w)(-v' + 4v^2 - 2w^2 + vw) \end{aligned} \quad (\text{A.11})$$

To render the system first order in derivatives on v , we introduce an auxiliary variable y , and postulate that the first derivatives of v and y be quadratic functions of v, w, y . Clearly, y is not unique, but is defined up to linear transformations on y of the form, $y \rightarrow sy + av + bw$, for s, a, b arbitrary real parameters. One readily establishes that such a system is given by,

$$\begin{aligned} v' &= 3v^2 - y(v - w) \\ w' &= 3w^2 + 2y(v - w) \\ y' &= 6v^2 + 3y^2 + 21vw + 9vy + 12wy \end{aligned} \quad (\text{A.12})$$

One verifies that, if w, v, y satisfy (A.12), then v, w satisfy (A.11). The AdS_5 solution has $v = w = 1/(3r)$ and $y \sim 1/r$, while the AdS_3 solution has $v = y = 0$ and $w = 1/(3r)$.

The system may be reduced to a single first order ordinary ODE by taking pairwise ratios v'/w' and y'/w' , and using the ratios $\alpha \equiv v/w$ and $\beta \equiv y/w$. The resulting equations are,

$$\begin{aligned} w \frac{d\alpha}{dw} &= \frac{(\alpha - 1)(3\alpha - \beta - 2\alpha\beta)}{3 + 2\beta(\alpha - 1)} \\ w \frac{d\beta}{dw} &= \frac{6\alpha^2 + 5\beta^2 + 21\alpha + 9\beta + 9\alpha\beta - 2\alpha\beta^2}{3 + 2\beta(\alpha - 1)} \end{aligned} \quad (\text{A.13})$$

Taking the ratio, we get a single first order ODE between α and β ,

$$\frac{d\alpha}{d\beta} = \frac{(\alpha - 1)(3\alpha - \beta - 2\alpha\beta)}{6\alpha^2 + 5\beta^2 + 21\alpha + 9\beta + 9\alpha\beta - 2\alpha\beta^2} \quad (\text{A.14})$$

The solution $\alpha = 1$ corresponds to AdS_5 , while $\alpha = \beta = 0$ corresponds to $AdS_3 \times T^2$. Equation (A.14) is of the Darboux type of order $m = 2$ according to [30], and of the Abel second kind type according to [31]. No general solutions are known for either type of equations, and (A.14) does not fit into any of the known categories of solvable special cases. Either way, we have not succeeded in integrating equation (A.14) analytically.

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